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A NOTE ON A THEOREM OF HEATH

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We propose a generalization of Heath's theorem that semi-metric spaces with point-countable bases are developable: A semi-metrizable space X is developable if (and only if) there is on it a σ-discrete family $\mathscr{C}=\bigcup_{m\in\mathbb{N}}\mathscr{C}_m$ of closed sets, interior-preserving over each member C of which is a countable family $\{\mathscr{D}_n(C)\colon n\in\mathbb{N}\}$ of collections of open sets such that if U is a neighbourhood of $\xi\in X$, then there are such a $\Gamma\in\mathscr{C}$ and such a $\nu\in\mathbb{N}$ that $\xi\in\Gamma$ and $\xi\in \operatorname{int}\bigcap(D\colon \xi\in D\in\mathscr{D}_\nu(\Gamma))\subset U$.

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developability
semi-metrizable spaces
Heath's condition for developability
interior-preserving open collections
point σ-interior-preserving collections of open sets

Heath's ingenious theorem that semi-metric spaces with point-countable bases are developable [3] has in its beauty an imperfection, in that its (sufficient) condition for developability is not necessary. We propose here, having been inspired by a characterization of developability in [6], to give a necessary and sufficient condition from which Heath's theorem readily follows (in a much more general form), thus providing it a simpler and more insightful proof.

Definitions 1. A collection $\mathscr{U}$ of open sets is *interior-preserving* [7] (cf. also Q-collections [4], [8]; and [5]) *over a set C*, if for every $x\in C$, $\bigcap\{U\colon x\in U\in\mathscr{U}\}$ is a *neighbourhood* of x. For every $x\in C$, we let $K(x,\mathscr{U})\equiv \operatorname{Int}\bigcap\{U\colon x\in U\in\mathscr{U}\}$ and call it the *cushion* at x with respect to $\mathscr{U}$.

Example 1. Point-finite collections are all interior-preserving (over the whole space).

Definition 2. A collection $\mathscr{U}$ of open sets on a topological space X is *point σ-interior-preserving*, if, for every $y\in X$, the subcollection $\mathscr{U}_y\equiv\{U\colon y\in U\in\mathscr{U}\}$ can be written as a *countable* family of collections each interior-preserving over all of X.

Example 2. Point countable collections are all point σ-interior-preserving.

Definitions 3. A T_1 space X is a *semi-metrizable space* [3] if on $X\times X$ there can be defined a *symmetric, non-negative* (real valued) function ρ such that, for all $A\subset X$, $x\in \mathrm{Cl}\, A \Leftrightarrow \rho(x,A)=0$[1]. For every $x\in X$ and $i\in\mathbb{N}$, we let

$$V_{x,i}\equiv\{y\in X: \rho(x,y)<1/i\}.$$

It is well known and can easily be demonstrated that $V_{x,i}$ is a *neighbourhood* of x.

Theorem 1. *A semi-metrizable space X is developable if (and only if) there is on it a σ-discrete family $\mathscr{C}=\bigcup_{m\in\mathbb{N}}\mathscr{C}_m$ of closed sets, interior-preserving over each member C of which is a countable family $\{\mathscr{D}_n(C): n\in\mathbb{N}\}$ of collections of open sets such that if U is a neighbourhood of $\xi\in X$, then there are such a $\Gamma\in\mathscr{C}$ and such a $\nu\in\mathbb{N}$ that $\xi\in\Gamma$ and $\xi\in K(\xi,\mathscr{D}_\nu(\Gamma))\subset U$.*

Proof. Let X be a semi-metrizable space. Let there be a σ-discrete family $\mathscr{C}$ of the description above. For every $x\in X$ and $l,m,n\in\mathbb{N}$, let

$$W(x;l,m,n)\equiv\begin{cases}\mathrm{Int}\ V_{x,l}\backslash\bigcup(\mathscr{C}_m\backslash\{C\})\cap K(x,\mathscr{D}_n(C)), & \text{if } x\in C\in\mathscr{C}_m;\\ \mathrm{Int}\ V_{x,l}\backslash\bigcup\mathscr{C}_m, & \text{if } x\notin\bigcup\mathscr{C}_m.\end{cases}$$

For every $l,m,n\in\mathbb{N}$, let

$$\mathscr{W}(l,m,n)\equiv\{W(x;l,m,n): x\in X\}.$$

Clearly, $\mathscr{W}(l,m,n)$ is a cover, whatever $l,m,n\in\mathbb{N}$. That

$$\mathscr{W}\equiv\{\mathscr{W}(l,m,n): l,m,n\in\mathbb{N}\}$$

is a development can be seen in the following.

Let U be a neighbourhood of $\xi\in X$. By hypothesis, there are $\mu,\nu\in\mathbb{N}$ such that $\xi\in\Gamma$, for some $\Gamma\in\mathscr{C}_\mu$, and $\xi\in K(\xi,\mathscr{D}_\nu(\Gamma))\subset U$. Clearly,

$$\mathrm{St}(\xi,\mathscr{W}(\lambda,\mu,\nu))\subset K(\xi,\mathscr{D}_\nu(\Gamma)),$$

if λ is *so* that $V_{\xi,\lambda}\subset K(\xi,\mathscr{D}_\nu(\Gamma))$ (consequent on which, if $\xi\in W(x;\lambda,\mu,\nu)$, $x\in K(\xi,\mathscr{D}_\nu(\Gamma))$ and $K(x,\mathscr{D}_\nu(\Gamma))=K(\xi,\mathscr{D}_\nu(\Gamma))$). □

Remarks 1. In view of Bennett's [2] theorem that quasi-developable spaces are developable if they are perfect, for the conclusion of Theorem 1, we do not require $\mathscr{W}(l,m,n)$ in the proof to be a *cover*. Consequently, we do not need to define $W(x;l,m,n)$ if $x\notin\bigcup\mathscr{C}_m$ and therefore $\mathscr{C}_m$, $m\in\mathbb{N}$, need only be discrete relative to $\bigcup\mathscr{C}_m$ in the hypothesis. If we take advantage of this relaxation of conditions, of course we need to have instead $W(x;l,m,n)$ defined as follows:

$$\mathrm{Int}\ V_{x,l}\backslash\mathrm{Cl}\bigcup(\mathscr{C}_m\backslash\{C\})\cap K(x,\mathscr{D}_n(C)),\quad \text{if } x\in C\in\mathscr{C}_m.$$

[1] In particular, $\rho(x,y)=0\Rightarrow x\in\mathrm{Cl}\{y\}\Rightarrow x=y$.

Theorem 2. *Semi-metrizable spaces with point σ-interior-preserving bases are developable.*

Proof. Let there be a point σ-interior-preserving base $\mathscr{B}$ on a semi-metrizable space X. Let X be well ordered. For every $y \in X$ and every $i, j \in \mathbb{N}$, let

$$C(y; i, j) \equiv \operatorname{Int} V_{y,i} \setminus \bigcup \{\operatorname{Int} V_{z,i} : z < y\} \setminus \bigcup \{\operatorname{Int} V_{x,j} : x \notin \operatorname{Int} V_{y,i}\}.$$

Clearly, the family

$$\mathscr{C} \equiv \bigcup \{\{C(y; i, j) : y \in X\} : i, j \in \mathbb{N}\}$$

is σ-discrete and closed. On the other hand, by hypothesis, $\mathscr{B}_y$, for every $y \in X$, can be written as a countable family of collections of open sets, each of which is interior-preserving over $C(y; i, j)$ whatever $i, j \in \mathbb{N}$. For our conclusion, we need only show that given $\xi \in X$ and $U \in \mathscr{B}_\xi$, there are $y \in X$, $i, j \in \mathbb{N}$ such that $\xi \in C(y; i, j)$ and $\xi \in B \subset U$, for some $B \in \mathscr{B}_y$. To this end, we note that (while, whatever $i \in \mathbb{N}$, there are some $j \in \mathbb{N}$ and some $y \in X$ such that $\xi \in C(y; i, j)$) if i is so that $V_{\xi,i} \subset U$, then $y \in U$ ensuring that $U \in \mathscr{B}_y$. $\square$

Remarks 2. In the same manner that the $\delta\theta$-base of Aull's [1] may replace the point countable bases in Heath's theorem for developability, point σ-interior preserving bases may be replaced, for the same purpose, by the following: A base $\mathscr{B} \equiv \bigcup_{k \in \mathbb{N}} \mathscr{B}^k$ such that if U is a neighbourhood of $x \in X$, there is (at least) one such $\kappa \in \mathbb{N}$ that $\mathscr{B}^\kappa_x \equiv \{B : x \in B \in \mathscr{B}^\kappa\}$

(i) can be written as a countable family of collections, each interior preserving over X, and

(ii) can count among its members one that is contained in U.

Noting that for each k, we can, instead of well ordering the whole space, well order only those points in the space at which (i) above is true and arrive at a $C^k(y; i, j)$ for each such y, we can see the proof of the above is *mutatis mutandis* good for our new purpose.

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